

Multiple-correction and Faster Approximation

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Abstract

In this paper, we formulate a new *multiple-correction method*. The goal is to accelerate the rate of convergence. In particular, we construct some sequences to approximate the Euler-Mascheroni and Landau constants, which are faster than the classical approximations in literature.

1 Introduction

Euler constant was first introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$(1.1) \quad \gamma(n) := \sum_{m=1}^n \frac{1}{m} - \ln n.$$

It is also known as the Euler-Mascheroni constant. There are many famous unsolved problems about the nature of this constant. For example, it is a long-standing open problem if it is a rational number. See e.g. the survey papers or books of Brent and Zimmermann [3], Dence and Dence [15], Havil [22] and Lagarias [23]. A good part of its mystery comes from the fact that the known algorithms converging to γ are not very fast, at least, when they are compared to similar algorithms for π and e .

The sequence $(\gamma(n))_{n \in \mathbb{N}}$ converges very slowly toward γ , like $(2n)^{-1}$. To evaluate it more accurately, we need to accelerate the convergence. This can be done using the Euler-Maclaurin summation formula, Stieltjes approach, exponential integral methods, Bessel function method, etc. See e.g. Gourdon and Sebah [19].

Up to now, many authors are preoccupied to improve its rate of convergence. See e.g. Chen and Mortici [11], DeTemple [16], Gavrea and Ivan [18], Lu [25, 26], Mortici [27], Mortici and

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Chen [31] and references therein. We list some main results as follows: as $n \rightarrow \infty$,

$$(1.2) \quad \sum_{m=1}^n \frac{1}{m} - \ln \left(n + \frac{1}{2} \right) = \gamma + O(n^{-2}), \quad (\text{DeTemple [16], 1993}),$$

$$(1.3) \quad \sum_{m=1}^n \frac{1}{m} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240} + \frac{107}{480}}{n^2 + n + \frac{97}{240}} = \gamma + O(n^{-6}), \quad (\text{Mortici [27], 2010}),$$

$$(1.4) \quad \sum_{m=1}^n \frac{1}{m} - \ln \rho(n) = \gamma + O(n^{-5}), \quad (\text{Chen and Mortici [11], 2012}),$$

where $\rho(n) = 1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}$. Recently, Mortici and Chen [31] provided a very interesting sequence

$$\begin{aligned} \nu(n) = & \sum_{m=1}^n \frac{1}{m} - \frac{1}{2} \ln \left(n^2 + n + \frac{1}{3} \right) \\ & - \left(\frac{-\frac{1}{180}}{(n^2 + n + \frac{1}{3})^2} + \frac{\frac{8}{2835}}{(n^2 + n + \frac{1}{3})^3} + \frac{\frac{5}{1512}}{(n^2 + n + \frac{1}{3})^4} + \frac{\frac{592}{93555}}{(n^2 + n + \frac{1}{3})^5} \right), \end{aligned}$$

and proved

$$(1.5) \quad \lim_{n \rightarrow \infty} n^{12} (\nu(n) - \gamma) = -\frac{796801}{43783740}.$$

Hence, the rate of convergence of the sequence $(\nu(n))_{n \in \mathbb{N}}$ is n^{-12} .

Let $R_1(n) = \frac{a_1}{n}$ and for $k \geq 2$

$$(1.6) \quad R_k(n) := \frac{a_1}{n + \frac{a_2 n}{n + \frac{a_3 n}{n + \frac{a_4 n}{\ddots \frac{a_k n}{n + a_k}}}}},$$

where $(a_1, a_2, a_4, a_6, a_8, a_{10}, a_{12}) = (\frac{1}{2}, \frac{1}{6}, \frac{3}{5}, \frac{79}{126}, \frac{7230}{6241}, \frac{4146631}{3833346}, \frac{306232774533}{179081182865})$, $a_{2k+1} = -a_{2k}$ for $1 \leq k \leq 6$, and

$$(1.7) \quad r_k(n) := \sum_{m=1}^n \frac{1}{m} - \ln n - R_k(n).$$

Lu [25] introduced a continued fraction method to investigate this problem, and showed

$$(1.8) \quad \frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}.$$

In fact, Lu [25] determined the constants a_1 to a_4 . Xu and You [39] continued Lu's work to find a_5, \dots, a_{13} with the help of *Mathematica* software, and obtained

$$(1.9) \quad \lim_{n \rightarrow \infty} n^{k+1} (r_k(n) - \gamma) = C'_k,$$

where $(C'_1, \dots, C'_{13}) = \left(-\frac{1}{12}, -\frac{1}{72}, \frac{1}{120}, \frac{1}{200}, -\frac{79}{25200}, -\frac{6241}{3175200}, \frac{241}{105840}, \frac{58081}{22018248}, -\frac{262445}{91974960}, -\frac{2755095121}{892586949408}, \frac{20169451}{3821257440}, \frac{406806753641401}{45071152103463200}, -\frac{71521421431}{5152068292800}\right)$. Moreover, they improved (1.8) to

$$(1.10) \quad C'_{10} \frac{1}{(n+1)^{11}} < \gamma - r_{10}(n) < C'_{10} \frac{1}{n^{11}},$$

$$(1.11) \quad C'_{11} \frac{1}{(n+1)^{12}} < r_{11}(n) - \gamma < C'_{11} \frac{1}{n^{12}}.$$

However, it seems difficult for us to find more constants a_k . One of the main reasons is due to the recursive algorithm. The other reason is that the parameter a_j appears many times in the coefficients of polynomials $P_l(x)$ and $Q_m(x)$, and this causes that expanding function $\frac{P_l(x)}{Q_m(x)}$ as power series in the terms of $1/x$ needs a huge of computations. To overcome this difficulty, the purpose of this paper is to formulate a new *multiple-correction method* to accelerate the convergence. In addition, we will use this method to study the sharp bounds for the constants of Landau.

The Landau's constants are defined for all integers $n \geq 0$ by

$$(1.12) \quad G(n) = \sum_{k=0}^n \frac{1}{16^k} \binom{2k}{k}^2.$$

The constants $G(n)$ are important in complex analysis. In 1913, Landau [24] proved that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function in the unit disc which satisfies $|f(z)| < 1$ for $|z| < 1$, then $|\sum_{k=0}^n a_k| \leq G(n)$, and that this bound is optimal. Landau [24] showed that

$$(1.13) \quad G(n) \sim \frac{1}{\pi} \ln n, (n \rightarrow \infty).$$

In 1930, Watson [37] obtained the following more precise asymptotic formula

$$(1.14) \quad G(n) \sim \frac{1}{\pi} \ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right), (n \rightarrow \infty),$$

where

$$(1.15) \quad c_0 = \frac{1}{\pi}(\gamma + 4 \ln 2) = 1.0662758532089143543 \dots$$

The work of Watson opened up a novel insight into the asymptotic behavior of the Landau sequences $(G(n))_{n \geq 0}$. Inspired by formula (1.14), many authors investigated the upper and lower bounds of $G(n)$. Some of the main results are listed as follows:

$$(1.16) \quad \frac{1}{\pi} \ln(n+1) + 1 \leq G(n) < \frac{1}{\pi} \ln(n+1) + c_0 \quad (n \geq 0), \quad (\text{Brutman [4], 1982}),$$

$$(1.17) \quad \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + c_0 < G(n) \leq \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + 1.0976 \quad (n \geq 0), \quad (\text{Falaleev [17], 1991}),$$

$$(1.18) \quad \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + c_0 < G(n) < \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192n}\right) + c_0 \quad (n \geq 1), \quad (\text{Mortici [30], 2011})$$

Very recently, Chen [9] presented the following better approximation to $G(n)$: as $n \rightarrow \infty$,

$$(1.19) \quad G(n) = c_0 + \frac{1}{\pi} \ln \left(n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} - \frac{2009}{184320(n + \frac{3}{4})^3} + \frac{2599153}{371589(n + \frac{3}{4})^5} \right) + O \left(\frac{1}{(n + \frac{3}{4})^8} \right),$$

and the better upper bound:

$$(1.20) \quad G(n) < c_0 + \frac{1}{\pi} \ln \left(n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} \right), (n \geq 0).$$

Another direction for developing the approximation to $G(n)$ was initiated by Cvijović and Klinowski [12], who established estimates for $G(n)$ in terms of the Psi(or Digamma) function $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$:

$$(1.21) \quad \frac{1}{\pi} \psi \left(n + \frac{5}{4} \right) + c_0 < G(n) < \frac{1}{\pi} \psi \left(n + \frac{5}{4} \right) + 1.0725, \quad (n \geq 0),$$

$$(1.22) \quad \frac{1}{\pi} \psi \left(n + \frac{3}{2} \right) + 0.9883 < G(n) < \frac{1}{\pi} \psi \left(n + \frac{3}{2} \right) + c_0, \quad (n \geq 0).$$

Since then, many authors have made significant contributions to sharper inequalities and asymptotic expansions for $G(n)$. See e.g. Alzer [1], Chen [8], Cvijović and Srivastava [14], Granath [21], Mortici [30], Nemes [32, 33], Popa [34], Popa and Secolean [35], Zhao [41], Gavrea and M. Ivan [18], Chen and Choi [5, 7], etc. To the best knowledge of authors, the latest upper bound is due to Chen [9], who proved

$$(1.23) \quad G(n) < c_0 + \frac{1}{\pi} \psi \left(n + \frac{5}{4} + \frac{1}{64(n + \frac{3}{4})} \right), (n \geq 0).$$

Here, the authors would like to thank Alzer, Chen, Choi, DeTemple, Granath, Lu, Mortici, etc., it is their important works that makes the present work becomes possible.

Notation. Throughout the paper, the notation $P_k(x)$ (or $Q_k(x)$) as usual denotes a polynomial of degree k in terms of x . The notation $\Psi(k; x)$ means a polynomial of degree k in terms of x with all of its non-zero coefficients being positive, which may be different at each occurrence. Notation $\Phi(k; x)$ denotes a polynomial of degree k in terms of x with the leading coefficient being equal to one, which may be different at different subsection.

2 Some Lemmas

The following lemma gives a method for measuring the rate of convergence, for the proof of which, see Mortici [27, 28].

Lemma 1. *If the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to zero and there exists the limit*

$$(2.1) \quad \lim_{n \rightarrow +\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty]$$

with $s > 1$, then

$$(2.2) \quad \lim_{n \rightarrow +\infty} n^{s-1} x_n = \frac{l}{s-1}.$$

In the study of Landau constants, we need to apply a so-called Brouncker's continued fraction formula.

Lemma 2. *For all integer $n \geq 0$, we have*

$$(2.3) \quad q(n) := \left(\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \right)^2 = \frac{4}{1 + 4n + \frac{1^2}{2 + 8n + \frac{3^2}{2 + 8n + \frac{5^2}{2 + 8n + \ddots}}}}.$$

In 1654 Lord William Brouncker found this remarkable fraction formula, when Brouncker and Wallis collaborated on the problem of squaring the circle. Formula (2.3) was not published by Brouncker himself, but first appeared in [36]. For a general n , Formula (2.3) follows from Entry 25 in Chapter 12 in Ramanujans notebook [2], which gives a more general continued fraction formula for quotients of gamma functions, and which have several proofs published by different authors.

Writing continued fractions in this way of (2.3) takes a lot of space. So instead we use the following shorthand notation

$$(2.4) \quad q(n) = \frac{4}{1 + 4n + \frac{1^2}{2 + 8n + \frac{3^2}{2 + 8n + \frac{5^2}{2 + 8n + \dots}}}},$$

and its k -th approximation $q_k(n)$ is defined by

$$(2.5) \quad q_1(n) = \frac{4}{1 + 4n},$$

$$(2.6) \quad q_k(n) = \frac{4}{1 + 4n + \frac{1^2}{2 + 8n + \frac{3^2}{2 + 8n + \dots \frac{(2k-3)^2}{2 + 8n}}}}, \quad (k \geq 2).$$

In the proof of our inequalities for the constants of Euler-Mascheroni and Landau, we also use the following simple inequality.

Lemma 3. *Let $f''(x)$ be a continuous function. If $f''(x) > 0$, then*

$$(2.7) \quad \int_a^{a+1} f(x) dx > f(a + 1/2).$$

Proof. By letting $x_0 = a + 1/2$ and Taylor's formula, we have

$$\begin{aligned}\int_a^{a+1} f(x)dx &= \int_a^{a+1} \left(f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\theta_x)(x - x_0)^2 \right) dx \\ &> \int_a^{a+1} (f(x_0) + f'(x_0)(x - x_0)) dx = f(a + 1/2).\end{aligned}$$

This completes the proof of Lemma 3. Also see Lemma 2 in Xu and You [39]. \square

3 Two Examples for Euler-Mascheroni Constant

In this section, to illustrate quickly and clearly the main ideas of this paper, we consider the simplest case of Euler-Mascheroni constant by using the correction-process again.

Example 1. We choose an initial-correction function $\theta_0(n)$ given by

$$(3.1) \quad \theta_0(n) = \frac{13 + 30n}{6(1 + 6n + 10n^2)},$$

and define

$$(3.2) \quad \nu_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \theta_0(n).$$

Applying Lemma 1, one can check(See Theorem 1 in [39] , or (1.9))

$$(3.3) \quad \lim_{n \rightarrow \infty} n^6 (\nu_0(n) - \nu_0(n+1)) = \frac{1}{40},$$

$$(3.4) \quad \lim_{n \rightarrow \infty} n^5 (\nu_0(n) - \gamma) = \frac{1}{200}.$$

By using the similar idea of Kummer's acceleration method and inserting the correction function $-\frac{1}{200}\frac{1}{n^5}$ in (3.2) again, one can use Lemma 1 again to show

$$(3.5) \quad \lim_{n \rightarrow \infty} n^6 \left(\nu_0(n) - \frac{1}{200} \frac{1}{n^5} - \gamma \right) = -\frac{773}{126000}.$$

Furthermore, we try to obtain an algorithm with a faster convergent rate by using $\Phi(5; n)$ instead of n^5 . To do that, let

$$(3.6) \quad \theta(n) = \frac{\frac{1}{200}}{\Phi(5; n)} = \frac{\frac{1}{200}}{n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0},$$

$$(3.7) \quad \nu(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \theta_0(n) - \theta(n).$$

First, we use the method of undetermined coefficients to find $a_j (0 \leq j \leq 4)$. By using the *Mathematica* software, we expand the difference $\nu(n) - \nu(n+1)$ into a power series in terms of n^{-1} :

$$(3.8) \quad \begin{aligned} & \nu(n) - \nu(n+1) \\ &= \frac{-773 + 630a_4}{21000} \frac{1}{n^7} + \frac{4033 + 1050a_3 - 3150a_4 - 1050a_4^2}{30000} \frac{1}{n^8} \\ & \quad + \frac{-37657 + 4500a_2 - 15750a_3 + 31500a_4 - 9000a_3a_4 + 15750a_4^2 + 4500a_4^3}{112500} \frac{1}{n^9} \\ & \quad + \frac{\varphi_{10}}{500000} \frac{1}{n^{10}} + \frac{\varphi_{11}}{4125000} \frac{1}{n^{11}} + \frac{\varphi_{12}}{75000000} \frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right), \end{aligned}$$

where

$$\begin{aligned} \varphi_{10} &= 350191 + 22500a_1 - 90000a_2 + 210000a_3 - 22500a_3^2 - 315000a_4 - 45000a_2a_4 \\ & \quad + 180000a_3a_4 - 210000a_4^2 + 67500a_3a_4^2 - 90000a_4^3 - 22500a_4^4, \\ \varphi_{11} &= -5465923 + 206250a_0 - 928125a_1 + 2475000a_2 - 4331250a_3 - 412500a_2a_3 \\ & \quad + 928125a_3^2 + 5197500a_4 - 412500a_1a_4 + 1856250a_2a_4 - 4950000a_3a_4 \\ & \quad + 618750a_3^2a_4 + 4331250a_4^2 + 618750a_2a_4^2 - 2784375a_3a_4^2 + 2475000a_4^3 \\ & \quad - 825000a_3a_4^3 + 928125a_4^4 + 206250a_4^5, \\ \varphi_{12} &= 175990871 - 20625000a_0 + 61875000a_1 - 123750000a_2 - 4125000a_2^2 \\ & \quad + 173250000a_3 - 8250000a_1a_3 + 41250000a_2a_3 - 61875000a_3^2 + 4125000a_3^3 \\ & \quad - 173250000a_4 - 8250000a_0a_4 + 41250000a_1a_4 - 123750000a_2a_4 + 247500000a_3a_4 \\ & \quad + 24750000a_2a_3a_4 - 61875000a_3^2a_4 - 173250000a_4^2 + 12375000a_1a_4^2 - 61875000a_2a_4^2 \\ & \quad + 185625000a_3a_4^2 - 24750000a_3^2a_4^2 - 123750000a_4^3 - 16500000a_2a_4^3 \\ & \quad + 82500000a_3a_4^3 - 61875000a_4^4 + 20625000a_3a_4^4 - 20625000a_4^5 - 4125000a_4^6. \end{aligned}$$

According to Lemma 1, we have five parameters a_4, a_3, a_2, a_1 and a_0 which produce the fastest convergence of the sequence from (3.8)

$$\begin{cases} -773 + 630a_4 = 0, \\ 4033 + 1050a_3 - 3150a_4 - 1050a_4^2 = 0 \\ -37657 + 4500a_2 - 15750a_3 + 31500a_4 - 9000a_3a_4 + 15750a_4^2 + 4500a_4^3 = 0 \\ \varphi_{10} = 0, \\ \varphi_{11} = 0, \end{cases}$$

namely if

$$(3.9) \quad \theta(n) = \frac{\frac{1}{200}}{n^5 + \frac{773}{630}n^4 + \frac{21361}{15876}n^3 + \frac{1348075}{2000376}n^2 - \frac{91207415}{252047376}n - \frac{178345771979}{1746688315680}}.$$

Thus, we get

$$(3.10) \quad \nu(n) - \nu(n+1) = -\frac{10992878936527}{160060165655040} \frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right).$$

We can apply another approach to find a_4, a_3, a_2, a_1 and a_0 step by step, which is achieved by using $n^5 + a_4n^4, n^5 + a_4n^4 + a_3n^3, n^5 + a_4n^4 + a_3n^3 + a_2n^2, n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_1n, n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_1n + a_0$ instead of $\Phi(5; n)$ in turn. For the reader's convenience, here we give an example to explain how *Mathematica* software generates $\nu(n) - \nu(n+1)$ into power series in the terms of $\frac{1}{n}$. For example, find a_3 . We manipulate *Mathematica* program

$$\begin{aligned} &\text{Normal}[\text{Series}[\left(-\frac{1}{n+1} + \log\left[1 + \frac{1}{n}\right] - \left(\frac{13+30n}{6(1+6n+10n^2)} + \frac{1/200}{n^5 + (773/630)n^4 + a_3n^3}\right)\right. \\ &\quad \left.+ \left(\frac{13+30n}{6(1+6n+10n^2)} + \frac{1/200}{n^5 + (773/630)n^4 + a_3n^3}\right) /. n \rightarrow n+1\right) /. n \rightarrow 1/x, \{x, 0, 10\}]] /. x \rightarrow 1/n \end{aligned}$$

to generate

$$(3.11) \quad \nu(n) - \nu(n+1) = \frac{-\frac{21361}{453600} + \frac{7a_3}{200}}{n^8} + \frac{\frac{9173092}{31255875} - \frac{3751a_3}{15750}}{n^9} + O\left(\frac{1}{n^{10}}\right).$$

By solving the equation $-\frac{21361}{453600} + \frac{7a_3}{200} = 0$, we also find $a_3 = \frac{21361}{15876}$. In what follows, we always use this approach.

By Lemma 1 again, we obtain finally

$$(3.12) \quad \lim_{n \rightarrow \infty} n^{11} (\nu(n) - \gamma) = -\frac{10992878936527}{1760661822205440}.$$

We observe that the above twice-correction improves the rate of convergence from n^{-6} to n^{-11} . \square

Remark 1. The main idea of twice-correction is that from $n^5, n^5 + a_4n^4, n^5 + a_4n^4 + a_3n^3 + a_2n^2, n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_1n$ to $n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_1n + a_0$, their approximations in turn become better and better.

Remark 2. It should be noted that once we find the exact values of the parameters a_4 to a_0 , it is not very difficult for us to check the formula (3.10) with the help of *Mathematica* software.

Example 2. We would like to give another example. Now we take the initial-correction function $\eta_0(n) = \frac{6n-1}{12n^2}$ (see Theorem 1.1 in Lu [25] or (1.9), which is found by the continued fraction method), and define

$$(3.13) \quad \mu_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n).$$

One may check by using Lemma 1

$$(3.14) \quad \lim_{n \rightarrow \infty} n^4 (\mu_0(n) - \gamma) = \frac{1}{120}.$$

Similarly, we insert a correction function $-\eta(n)$ in (3.13) again, which has the form of

$$(3.15) \quad \eta(n) = \frac{\frac{1}{120}}{n^4 + b_3 n^3 + b_2 n^2 + b_1 n + b_0}.$$

Applying the same method as Example 1, we can find $(b_3, b_2, b_1, b_0) = (0, \frac{10}{21}, 0, -\frac{241}{882})$. The details is omitted here. Now we define

$$(3.16) \quad \mu(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \eta(n),$$

where

$$(3.17) \quad \eta_0(n) = \frac{6n-1}{12n^2} \quad \text{and} \quad \eta(n) = \frac{\frac{1}{120}}{n^4 + \frac{10}{21}n^2 - \frac{241}{882}}.$$

By using *Mathematica* software and Lemma 1, we can attain

$$(3.18) \quad \mu(n) - \mu(n+1) = -\frac{13775}{305613} \frac{1}{n^{11}} + O\left(\frac{1}{n^{12}}\right),$$

$$(3.19) \quad \lim_{n \rightarrow \infty} n^{10} (\mu(n) - \gamma) = -\frac{13775}{3056130}.$$

Remark 3. We observe that the above twice-correction improves the rate of convergence from n^{-4} to n^{-10} , which is the desired result. However, it is interesting to note that both b_3 and b_1 equal zero. The reason of why inserting the sub-correction term $b_3 n^3$ (or $b_1 n$) does not improves the rate of convergence (i.e. compare $n^4 + b_3 n^3$ with n^4 , or $n^4 + b_3 n^3 + b_2 n^2 + b_1 n$ with $n^4 + b_3 n^3 + b_2 n^2$) may be that the function n^3 (or n) *changes* too rapidly when n tends to infinity. Fortunately, these losses are made up by the sub-correction terms $b_2 n^2$ and b_0 .

More precisely, we will improve (3.19), and prove the following double-sides inequalities.

Theorem 1. *Let $\mu(n)$ be defined by (3.16). Then for all positive integer n , we have*

$$(3.20) \quad \frac{13775}{3056130} \frac{1}{(n + \frac{3}{4})^{10}} < \gamma - \mu(n) < \frac{13775}{3056130} \frac{1}{(n - \frac{1}{4})^{10}}.$$

Remark 4. In fact, Theorem 1 implies that $\mu(n)$ is a strictly increasing function of n .

Proof. It follows from (3.16)

$$(3.21) \quad \mu(n) - \mu(n+1) = -\frac{1}{n+1} + \ln(1 + \frac{1}{n}) + \eta_0(n+1) + \eta(n+1) - \eta_0(n) - \eta(n).$$

We write $D = \frac{13775}{27783}$, and define for $x \geq 1$

$$(3.22) \quad -\omega(x) = -\frac{1}{x+1} + \ln(1 + \frac{1}{x}) + \eta_0(x+1) + \eta(x+1) - \eta_0(x) - \eta(x).$$

By *Mathematica* software, it is not difficult to check

(3.23)

$$\begin{aligned}
& -\omega'(x) - \frac{D}{(x + \frac{1}{4})^{12}} \\
& = -\frac{\Psi_1(20; x)(x-1) + 4032098201877889488940287625}{277830x^3(1+x)^3(1+4x)^{12}(-241+420x^2+882x^4)^2(1061+4368x+5712x^2+3528x^3+882x^4)^2} \\
& < 0,
\end{aligned}$$

and

(3.24)

$$\begin{aligned}
& -\omega'(x) - \frac{D}{(x + \frac{3}{4})^{12}} \\
& = \frac{\Psi_2(20; x)(x-1) + 51726219719747325679290363431}{277830x^3(1+x)^3(3+4x)^{12}(-241+420x^2+882x^4)^2(1061+4368x+5712x^2+3528x^3+882x^4)^2} \\
& > 0.
\end{aligned}$$

Note that $\omega(\infty) = 0$. From (3.23) and Lemma 3, one has

(3.25)

$$\begin{aligned}
\omega(n) &= \int_n^\infty -\omega'(x)dx < D \int_n^\infty \frac{dx}{(x + \frac{1}{4})^{12}} \\
&= \frac{D}{11} \frac{1}{(n + \frac{1}{4})^{11}} < \frac{D}{11} \int_{n-\frac{1}{4}}^{n+\frac{3}{4}} \frac{dx}{x^{11}}.
\end{aligned}$$

Note that $\mu(\infty) = \gamma$. Combining (3.21), (3.22) and (3.25), we have

(3.26)

$$\begin{aligned}
\gamma - \mu(n) &= \sum_{m=n}^\infty (\mu(m+1) - \mu(m)) < \frac{D}{11} \sum_{m=n}^\infty \int_{m-\frac{1}{4}}^{m+\frac{3}{4}} \frac{dx}{x^{11}} \\
&= \frac{D}{11} \int_{n-\frac{1}{4}}^\infty \frac{dx}{x^{11}} = \frac{D}{110} \frac{1}{(n - \frac{1}{4})^{10}}.
\end{aligned}$$

This finishes the proof of right-hand inequality in (3.20). Similarly, it follows from (3.24)

(3.27)

$$\begin{aligned}
\omega(n) &= \int_n^\infty -\omega'(x)dx > D \int_n^\infty \frac{dx}{(x + \frac{3}{4})^{12}} \\
&= \frac{D}{11} \frac{1}{(n + \frac{3}{4})^{11}} > \frac{D}{11} \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{dx}{x^{11}}.
\end{aligned}$$

Finally, by (3.21), (3.22) and (3.27), one has

(3.28)

$$\begin{aligned}
\gamma - \mu(n) &= \sum_{m=n}^\infty (\mu(m+1) - \mu(m)) > \frac{D}{11} \sum_{m=n}^\infty \int_{m+\frac{3}{4}}^{m+\frac{7}{4}} \frac{dx}{x^{11}} \\
&= \frac{D}{11} \int_{n+\frac{3}{4}}^\infty \frac{dx}{x^{11}} = \frac{D}{110} \frac{1}{(n + \frac{3}{4})^{10}}.
\end{aligned}$$

This completes the proof of Theorem 1. \square

4 The multiple-correction method

Based on the work of Section 2, we will formulate a new *multiple-correction method* to study faster approximation problem for the constants of Euler-Mascheroni and Landau.

Let $(v(n))_{n \geq 1}$ be a sequence to be approximated. Throughout the paper, we always assume that the following three conditions hold.

Condition (i). The initial-correction function $\eta_0(n)$ satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} (v(n) - \eta_0(n)) &= 0, \\ \lim_{n \rightarrow \infty} n^{l_0} (v(n) - v(n+1) - \eta_0(n) + \eta_0(n+1)) &= C_0 \neq 0, \end{aligned}$$

with some a positive integer $l_0 \geq 2$.

Condition (ii). The k -th correction function $\eta_k(n)$ has the form of $-\frac{C_{k-1}}{\Phi_k(l_{k-1}; n)}$, where

$$\lim_{n \rightarrow \infty} n^{l_{k-1}} \left(v(n) - v(n+1) - \sum_{j=0}^{k-1} (\eta_j(n) - \eta_j(n+1)) \right) = C_{k-1} \neq 0,$$

Condition (iii). The difference $(v(1/x) - v(1/x+1) - \eta_0(1/x) + \eta_0(1/x+1))$ is an analytic function in a neighborhood of point $x = 0$.

4.1 Euler-Mascheroni Constant

(Step 1) The initial-correction. We choose $\eta_0(n) = 0$, and let

$$(4.1) \quad \nu_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n.$$

By lemma 1, it is not difficult to prove that

$$(4.2) \quad \lim_{n \rightarrow \infty} n^2 (\nu_0(n) - \nu_0(n+1)) = \frac{1}{2},$$

$$(4.3) \quad \lim_{n \rightarrow \infty} n (\nu_0(n) - \gamma) = \frac{1}{2} =: C_0.$$

(Step 2) The first-correction. We let

$$(4.4) \quad \eta_1(n) = \frac{C_0}{\Phi_1(1; n)} = \frac{\frac{1}{2}}{n + b_{(1,0)}},$$

and define

$$(4.5) \quad \nu_1(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n).$$

By the same method as (3.11), we find $b_{(1,0)} = \frac{1}{6}$. Applying Lemma 1 again, one has

$$(4.6) \quad \lim_{n \rightarrow \infty} n^4 (\nu_1(n) - \nu_1(n+1)) = -\frac{1}{24},$$

$$(4.7) \quad \lim_{n \rightarrow \infty} n^3 (\nu_1(n) - \gamma) = -\frac{1}{72} := C_1.$$

(Step 3) The second-correction. Similarly, we set the second-correction function in the form of $\eta_2(n) = \frac{C_1}{\Phi_2(3;n)}$, and define

$$(4.8) \quad \nu_2(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n).$$

By using similar approach of (3.11), we can find

$$(4.9) \quad \eta_2(n) = \frac{-\frac{1}{72}}{n^3 + \frac{23}{30}n^2 + \frac{14}{25}n + \frac{1333}{10500}} = \frac{-\frac{1}{72}}{(n + \frac{23}{90})^3 + \frac{983}{2700}(n + \frac{23}{90}) + \frac{2197}{127575}}.$$

By Lemma 1, one can obtain

$$(4.10) \quad \lim_{n \rightarrow \infty} n^8 (\nu_2(n) - \nu_2(n+1)) = -\frac{7061}{5400},$$

$$(4.11) \quad \lim_{n \rightarrow \infty} n^7 (\nu_2(n) - \gamma) = -\frac{7061}{3780000} := C_2.$$

(Step 4) The third-correction. We set $\eta_3(n) = \frac{C_2}{\Phi_3(7;n)}$, and define

$$(4.12) \quad \nu_3(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n) - \eta_3(n).$$

By using *Mathematica* software, we can find

$$(4.13) \quad \begin{aligned} \Phi_3(7;n) = & n^7 + \frac{126901}{70610}n^6 + \frac{302657774122}{78525910575}n^5 + \frac{203050524517143511}{52278737145178500}n^4 \\ & + \frac{4586975399311716291806}{2713180197918474605475}n^3 + \frac{139644786102811402696525439}{298861139889036647352440010}n^2 \\ & + \frac{1335669056713380727335512329403306}{243734857761374337083369364175455}n \\ & + \frac{1768275723433572920281319954725767947341}{1032607098391838516487402648265732653000}. \end{aligned}$$

Now by Lemma 1 again, we obtain

Theorem 2. Let $\nu_3(n)$ be defined by (4.12). Then we have

$$(4.14) \quad \lim_{n \rightarrow \infty} n^{16} (\nu_3(n) - \nu_3(n+1)) = 15C_3,$$

and

$$(4.15) \quad \lim_{n \rightarrow \infty} n^{15} (\nu_3(n) - \gamma) = -\frac{6044981017774921659252823535814990412377703}{102460439337930176798462527774167322493925000} := C_3.$$

Remark 5. It could be imagined that if we apply the correction-process many times, then, we can obtain k th-correction sequence

$$(4.16) \quad \nu_k(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \sum_{j=1}^k \frac{C_{j-1}}{\Phi_j(l_{j-1}; n)}$$

with the rate of convergence $\frac{1}{n^{l_k}}$, here $l_k \geq 2l_{k-1} + 1$, i.e.

$$(4.17) \quad \lim_{n \rightarrow \infty} n^{l_k} (\nu_k(n) - \gamma) = C_k \neq 0,$$

$$(4.18) \quad \gamma = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \sum_{j=1}^k \frac{C_{j-1}}{\Phi_j(l_{j-1}; n)} + O\left(\frac{1}{n^{l_k}}\right).$$

Remark 6. For comparison, the result $\nu_1(n)$ in Theorem 2 is the same as $r_2(n)$ in (1.9), and $\lim_{n \rightarrow \infty} \frac{\nu_2(n) - \gamma}{r_6(n) - \gamma} = \frac{C_2}{C_6} = 0.950367 \dots < 1$. Theoretically at least, for a large n the above formula may reduce or eliminate numerically computations compared with Euler-Maclaurin summation formula. For example, if we take $n = 2^{15} = 32768$ in Theorem 2, then $-1.09418 \cdot 10^{-69} < \nu_3(n) - \gamma < 0$.

Remark 7. We can take different initial-correction function to find some other simple faster approximations. For example, we choose the initial-correction function $\eta_0(n) = -\ln n + \frac{1}{2} \ln(n^2 + n + \frac{1}{3})$, see, Chen and Li [10]. By Lemma 1, it is not very difficult for us to check that

$$(4.19) \quad \nu_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) = \sum_{m=1}^n \frac{1}{m} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right),$$

$$(4.20) \quad \lim_{n \rightarrow \infty} n^4 (\nu_0(n) - \gamma) = -\frac{1}{180} =: C_0.$$

Let

$$(4.21) \quad \eta_1(n) = \frac{C_0}{\Phi_1(4; n)} \quad \text{and} \quad \eta_2(n) = \frac{C_1}{\Phi_2(10; n)},$$

where $C_1 = \frac{457528}{123773265}$,

$$(4.22) \quad \eta_1(n) = \left(n + \frac{1}{2}\right)^4 + \frac{85}{126} \left(n + \frac{1}{2}\right)^2 - \frac{18287}{63504},$$

$$(4.23) \quad \begin{aligned} \eta_2(n) = & \left(n + \frac{1}{2}\right)^{10} + \frac{28038237821}{5995446912} \left(n + \frac{1}{2}\right)^8 + \frac{11612938185382451401}{35945383674610335744} \left(n + \frac{1}{2}\right)^6 \\ & + \frac{163544744039006129564874642269}{8307589279805355451415003136} \left(n + \frac{1}{2}\right)^4 \\ & - \frac{2762081970439756978947606523226093660107}{15021373006058621011789214048600457216} \left(n + \frac{1}{2}\right)^2 \\ & + \frac{2771007475606973680958970352585491511233080640189551}{1350897666047614749541649384438829437061829754880}. \end{aligned}$$

Now define

$$(4.24) \quad \nu_2(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n).$$

By using Lemma 1, one may check

$$(4.25) \quad \lim_{n \rightarrow \infty} n^{22} (\nu_2(n) - \gamma) = C_2,$$

where

$$(4.26) \quad C_2 = \frac{1864841554154123790589398711158437373230857062719102654146029}{18653565767176841210967548892254397636853629986414159462400}.$$

Some other interesting correction functions can be found in Gourdon and Sebah [20].

4.2 Landau Constants

(Step 1) The initial-correction. Let c_0 be defined by (1.15). Motivated by inequalities (1.17) and (1.18), we choose $\eta_0(n) = \frac{1}{\pi} \ln(n + \frac{3}{4}) + c_0$, and define

$$(4.27) \quad u_0(n) = G(n) - \eta_0(n) = G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0.$$

Now we consider the difference $u_0(n) - u_0(n+1)$. It follows immediately from (4.27)

$$(4.28) \quad u_0(n) - u_0(n+1) = G(n) - G(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}).$$

First, from the duplication formula (Legendre, 1809)

$$(4.29) \quad 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} \Gamma(2z),$$

one can prove

$$(4.30) \quad G(n) - G(n-1) = \frac{(\Gamma(2n+1))^2}{16^2 (\Gamma(n+1))^4} = \left(\frac{(2n)!}{4^n (n!)^2} \right)^2 = \frac{1}{\pi} q(n),$$

where $q(n)$ is defined by (2.3). Also see p.739 in Granath [21] or p.306 in Chen [9]. By (4.28) and (4.30), one has

$$(4.31) \quad u_0(n) - u_0(n+1) = -\frac{1}{\pi} q(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}).$$

From Lemma 2 and (2.6), on one hand, it can be observed that for all positive integer j , one has

$$(4.32) \quad q_2(n) < q_4(n) < \cdots < q_{2j}(n) < q(n) < q_{2j+1}(n) < \cdots < q_3(n) < q_1(n).$$

On the other hand, we can check by using *Mathematica* software

$$(4.33) \quad q_9(n) - q_8(n) = O\left(\frac{1}{n^{17}}\right).$$

Combining (4.32) and (4.33) gives

$$(4.34) \quad q(n+1) = q_8(n+1) + O\left(\frac{1}{n^{16}}\right).$$

By using the *Mathematica* software, we expand $q_8(n+1)$ into a power series in terms of n^{-1} . Noting formula (4.34), we obtain

$$(4.35) \quad \begin{aligned} q(n+1) = & q_8(n+1) + O\left(\frac{1}{n^{16}}\right) \\ = & \frac{1}{n} - \frac{5}{4} \frac{1}{n^2} + \frac{49}{32} \frac{1}{n^3} - \frac{235}{128} \frac{1}{n^4} + \frac{4411}{2048} \frac{1}{n^5} - \frac{20275}{8192} \frac{1}{n^6} + \frac{183077}{65536} \frac{1}{n^7} \\ & - \frac{815195}{262144} \frac{1}{n^8} + \frac{28754131}{8388608} \frac{1}{n^9} - \frac{125799895}{33554432} \frac{1}{n^{10}} + \frac{1091975567}{268435456} \frac{1}{n^{11}} \\ & - \frac{4702048685}{1073741824} \frac{1}{n^{12}} + \frac{80679143663}{17179869184} \frac{1}{n^{13}} - \frac{346250976095}{68719476736} \frac{1}{n^{14}} \\ & + \frac{2947620308941}{549755813888} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right). \end{aligned}$$

The above expression is also used in the first and second-correction below. In addition, it is not difficult to obtain

$$(4.36) \quad -\ln\left(n + \frac{3}{4}\right) + \ln\left(n + \frac{7}{4}\right) = \frac{1}{n} - \frac{5}{4} \frac{1}{n^2} + \frac{79}{48} \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Inserting (4.35) and (4.36) into (4.31) yields

$$(4.37) \quad u_0(n) - u_0(n+1) = \frac{11}{96\pi} \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Note that the inequalities (1.18) implies $u_0(\infty) = 0$. Applying Lemma 1, we obtain

$$(4.38) \quad \lim_{n \rightarrow \infty} n^2 u_0(n) = \frac{11}{192\pi} = C_0.$$

(Step 2) The first-correction. We let

$$(4.39) \quad \eta_1(n) = \frac{C_0}{\Phi_1(2; n)} = \frac{C_0}{n^2 + a_1 n + a_0},$$

and define

$$(4.40) \quad u_1(n) = G(n) - \eta_0(n) - \eta_1(n).$$

Hence

$$(4.41) \quad \begin{aligned} u_1(n) - u_1(n+1) &= (u_0(n) - \eta_1(n)) - (u_0(n+1) - \eta_1(n+1)) \\ &= (u_0(n) - u_0(n+1)) - (\eta_1(n) - \eta_1(n+1)). \end{aligned}$$

Note that the first term of (4.41) can be treated by the same method in (step 1). Here we only need to replace (4.36) by the following more accurate power series expansion

$$(4.42) \quad \begin{aligned} -\ln(n + \frac{3}{4}) + \ln(n + \frac{7}{4}) &= \frac{1}{n} - \frac{5}{4} \frac{1}{n^2} + \frac{79}{48} \frac{1}{n^3} - \frac{145}{64} \frac{1}{n^4} + \frac{4141}{1280} \frac{1}{n^5} - \frac{14615}{3072} \frac{1}{n^6} \\ &\quad + \frac{205339}{28672} \frac{1}{n^7} - \frac{179945}{16384} \frac{1}{n^8} + \frac{10083481}{589824} \frac{1}{n^9} - \frac{7060405}{262144} \frac{1}{n^{10}} \\ &\quad + \frac{494287399}{11534336} \frac{1}{n^{11}} - \frac{865047235}{12582912} \frac{1}{n^{12}} + \frac{24221854021}{218103808} \frac{1}{n^{13}} \\ &\quad - \frac{84777286235}{469762048} \frac{1}{n^{14}} + \frac{1186886790259}{4026531840} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right). \end{aligned}$$

By applying *Mathematica* software again, we have

$$(4.43) \quad \begin{aligned} &\frac{1}{\Phi_1(2; n)} - \frac{1}{\Phi_1(2; n+1)} \\ &= \frac{2}{n^3} + \frac{-3 - 3a_1}{n^4} + \frac{4 + 6a_1 + 4a_1^2 - 4a_0}{n^5} \\ &\quad + \frac{-5 - 10a_1 - 10a_1^2 - 5a_1^3 + 10a_0 + 10a_1a_0}{n^6} \\ &\quad + \frac{6 + 15a_1 + 20a_1^2 + 15a_1^3 + 6a_1^4 - 20a_0 - 30a_1a_0 - 18a_1^2a_0 + 6a_0^2}{n^7} + O\left(\frac{1}{n^8}\right). \end{aligned}$$

Now combining (4.41), (4.31), (4.35), (4.42) and (4.43), and performing some simplifications, we can obtain

$$(4.44) \quad \begin{aligned} \pi(u_1(n) - u_1(n+1)) &= \frac{\frac{235}{128} + \frac{-134+11a_1}{64}}{n^4} \\ &\quad + \frac{-\frac{4411}{2048} + \frac{11543-1320a_1-880a_1^2+880a_0}{3840}}{n^5} \\ &\quad + \frac{\frac{20275}{8192} + \frac{5(-2747+352a_1+352a_1^2+176a_1^3-352a_0-352a_1a_0)}{3072}}{n^6} \\ &\quad + \frac{-\frac{183077}{65536} + \frac{\sigma}{86016}}{n^7} + O\left(\frac{1}{n^8}\right), \end{aligned}$$

where

$$\sigma = 586449 - 73920a_1 - 98560a_1^2 - 73920a_1^3 - 29568a_1^4 + 98560a_0 + 147840a_1a_0 + 88704a_1^2a_0 - 29568a_0^2.$$

The fastest sequence $(u_1(n))_{n \geq 1}$ is obtained when the first two coefficients of this power series vanish. In this case

$$(4.45) \quad a_1 = \frac{3}{2}, \quad a_0 = \frac{5501}{7040},$$

thus

$$u_1(n) - u_1(n+1) = \frac{89684299}{3027763200\pi} \frac{1}{n^7} + O\left(\frac{1}{n^8}\right).$$

Finally, by using Lemma 1, one has

$$(4.46) \quad \lim_{n \rightarrow \infty} n^6 u_1(n) = \frac{89684299}{18166579200\pi} = C_1.$$

(Step 3) The second-correction. We let

$$(4.47) \quad \eta_2(n) = \frac{C_1}{\Phi_2(6; n)} = \frac{C_1}{n^6 + b_5 n + b_4 n^4 + b_3 n^3 + b_2 n^2 + b_1 n + b_0},$$

and define

$$(4.48) \quad u_2(n) = G(n) - \eta_0(n) - \eta_1(n) - \eta_2(n).$$

Thus

$$(4.49) \quad u_2(n) - u_2(n+1) = (u_0(n) - u_0(n+1)) - (\eta_1(n) + \eta_2(n) - \eta_1(n+1) - \eta_2(n+1)).$$

We use (4.35) and (4.42) to expand $u_0(n) - u_0(n+1)$ into a power series as in terms of n^{-1} . In addition, as mentioned already in Section 3, one can use a similar *Mathematica* program in Example 1 to find b_5, b_4, b_3, b_2, b_1 and b_0 in turn. Here we omit the details. We write

$$(4.50) \quad C_2 = -\frac{5691942495934169497683736629269380931519449}{65873649616252391923660120676946385934745600\pi}.$$

By using Lemma 1 again, it is not very difficult for us to check the following assertion.

Theorem 3. *Let c_0, C_2 be defined by (1.15) and (4.50) respectively, and*

$$(4.51) \quad u_2(n) := G(n) - \left(\frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + c_0 + \frac{\frac{11}{192\pi}}{\Phi_1(2; n)} + \frac{\frac{89684299}{18166579200\pi}}{\Phi_2(6; n)} \right),$$

where

$$(4.52) \quad \Phi_1(2; n) = \left(n + \frac{3}{4}\right)^2 + \frac{1541}{7040},$$

$$(4.53) \quad \begin{aligned} \Phi_2(6; n) = & \left(n + \frac{3}{4}\right)^6 + \frac{1092000370209}{631377464960} \left(n + \frac{3}{4}\right)^4 - \frac{111862508515629162375}{181198865117870921728} \left(n + \frac{3}{4}\right)^2 \\ & + \frac{1824588073050833974528912179250963}{540823069619183303269309779804160}. \end{aligned}$$

Then we have

$$(4.54) \quad \lim_{n \rightarrow \infty} n^{15} (u_2(n) - u_2(n+1)) = 14C_2,$$

$$(4.55) \quad \lim_{n \rightarrow \infty} n^{14} u_2(n) = C_2.$$

Remark 8. It should be stressed that that a “good” initial-correction is very important for us to accelerate the convergence. In addition, one may study analogous question by choosing different initial-correction.

The following Theorem tells us how to improve (1.17) and (1.20).

Theorem 4. *Let c_0 be defined by (1.15). Then for all integer $n \geq 0$, we have*

$$(4.56) \quad \frac{C_1}{(n + \frac{3}{2})^6} < G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0 - \frac{\frac{11}{192\pi}}{(n + \frac{3}{4})^2 + \frac{1541}{7040}} < \frac{C_1}{(n + \frac{1}{2})^6},$$

where $C_1 = \frac{89684299}{18166579200\pi}$.

Remark 9. In fact, Theorem 4 implies that $u_1(n)$ is a strictly decreasing function of n .

Proof. Although the method used in this section is very similar to that in proof of Theorem 1, we would like to give a full proof for the sake of completeness. First, we can see that the inequalities (4.56) are true for $n = 0$. Hence, in the following we only need to prove that these inequalities are also true for $n \geq 1$. To this end, let

$$(4.57) \quad u_1(n) = G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0 - \frac{\frac{11}{192\pi}}{\Phi_1(2; n)},$$

it follows easily from (4.30)

$$(4.58) \quad \begin{aligned} u_1(n) - u_1(n+1) = & -\frac{1}{\pi} q(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - \frac{\frac{11}{192\pi}}{\Phi_1(2; n)} \\ & + \frac{1}{\pi} \ln(n + \frac{7}{4}) + \frac{\frac{11}{192\pi}}{\Phi_1(2; n+1)}. \end{aligned}$$

Let

$$(4.59) \quad f(x) = -\frac{1}{\pi} q_6(x+1) - \frac{1}{\pi} \ln(x + \frac{3}{4}) - \frac{\frac{11}{192\pi}}{\Phi_1(2; x)} + \frac{1}{\pi} \ln(x + \frac{7}{4}) + \frac{\frac{11}{192\pi}}{\Phi_1(2; x+1)},$$

$$(4.60) \quad g(x) = -\frac{1}{\pi} q_5(x+1) - \frac{1}{\pi} \ln(x + \frac{3}{4}) - \frac{\frac{11}{192\pi}}{\Phi_1(2; x)} + \frac{1}{\pi} \ln(x + \frac{7}{4}) + \frac{\frac{11}{192\pi}}{\Phi_1(2; x+1)}.$$

From (4.32) and (4.60), one has

$$(4.61) \quad g(n) < u_1(n) - u_1(n+1) < f(n).$$

Firstly, we give the lower bound for $g(n)$, and the upper bound for $f(n)$, respectively. We set $D_1 = \frac{89684299}{432537600}$. By using the *Mathematica* software, we easily obtain

$$(4.62) \quad -f'(x) - \frac{D_1}{\pi(x+1)^8} = -\frac{1}{\pi} \frac{\Psi_1(21; n)}{\Psi_2(30; n)} < 0.$$

Noting $f(+\infty) = 0$ and utilizing (4.62) and Lemma 3, one has

$$(4.63) \quad \begin{aligned} f(n) &= -\int_n^\infty f'(x)dx < \int_n^\infty \frac{D_1}{\pi(x+1)^8}dx = \frac{D_1}{7\pi} \frac{1}{(n+1)^7} \\ &< \frac{D_1}{7\pi} \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{1}{x^7}dx. \end{aligned}$$

Similarly, we can check

$$(4.64) \quad -g'(x) - \frac{D_1}{\pi(x+\frac{3}{2})^8} = \frac{1}{\pi} \frac{\Psi_3(19; n)}{\Psi_4(28; n)} > 0.$$

Applying $g(+\infty) = 0$ and (4.64), we obtain

$$(4.65) \quad \begin{aligned} g(n) &= -\int_n^\infty g'(x)dx > \int_n^\infty \frac{D_1}{\pi(x+\frac{3}{2})^8}dx = \frac{D_1}{7\pi} \frac{1}{(n+\frac{3}{2})^7} \\ &> \frac{D_1}{7\pi} \int_{n+\frac{3}{2}}^{n+\frac{5}{2}} \frac{1}{x^7}dx. \end{aligned}$$

On the other hand, from $u_1(\infty) = 0$ and (4.63), we have

$$(4.66) \quad \begin{aligned} u_1(n) &= \sum_{m=n}^\infty (u_1(m) - u_1(m+1)) < \sum_{m=n}^\infty \frac{D_1}{7\pi} \int_{m+\frac{1}{2}}^{m+\frac{3}{2}} \frac{1}{x^7}dx \\ &= \frac{D_1}{7\pi} \int_{n+\frac{1}{2}}^\infty \frac{1}{x^7}dx = \frac{D_1}{42\pi} \frac{1}{(n+\frac{1}{2})^6}. \end{aligned}$$

Similarly, it follows from (4.65)

$$(4.67) \quad \begin{aligned} u_1(n) &= \sum_{m=n}^\infty (u_1(m) - u_1(m+1)) > \sum_{m=n}^\infty \frac{D_1}{7\pi} \int_{m+\frac{3}{2}}^{m+\frac{5}{2}} \frac{1}{x^7}dx \\ &= \frac{D_1}{7\pi} \int_{n+\frac{3}{2}}^\infty \frac{1}{x^7}dx = \frac{D_1}{42\pi} \frac{1}{(n+\frac{3}{2})^6}. \end{aligned}$$

This completes the proof of Theorem 4. □

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